Spheres

Isn't That Spatial?

Lesson 36-1 Surface Area of Spheres

ACTIVITY 36

My Notes

Learning Targets:

- Solve problems using properties of spheres.
- Solve problems by finding the surface area of a sphere.

SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Interactive Word Wall

"Cavalieri's Kitchen Wows Customers with Volumes of Food and Fun!" was the headline in the latest edition of *Food Critics* magazine. When asked the secret of the restaurant's success, head chef and owner Greta stated, "I combine my math background with my love of cooking to create interesting, appealing, and delicious food."

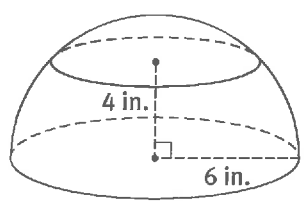
One of Greta's specialties is her ability to work with spherically shaped designs. A sphere is a three-dimensional shape such that all points are an equal distance from a fixed point. The fixed point is the center of the sphere and the distance from the fixed point is the radius of the sphere.

MATH TERMS

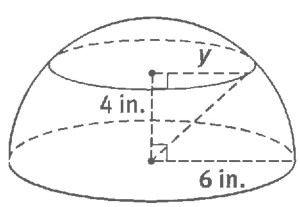
A **hemisphere** is one-half of a sphere.

Example A

Greta is preparing a cake in the shape of a ***hemisphere***, as shown below. She plans to wrap a piece of licorice around the cake, 4 inches up from the base. Given that the radius of the hemisphere is 6 inches, what is the total length of licorice needed?



Step 1: Use the Pythagorean Theorem to find the radius of the ring, y.



$$4^2 + y^2 = 6^2$$
$$y^2 = 20$$
$$y = \sqrt{20}$$

Step 2: To find the distance around the cake, use the circumference formula for a circle.

$C = 2\pi r$

$C = 2\pi\sqrt{20}$

$C \approx 28.1$ in.

Solution: The length of licorice needed is 28.1 in.

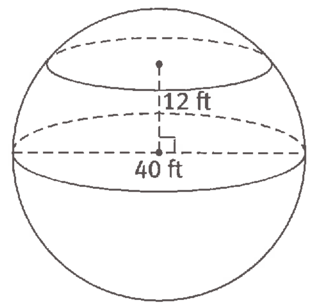
My Notes

Surface area of a sphere
$A = 4\pi r^2$

Try These A

a. The diameter of a sphere is 40 ft. A plane, parallel to the center of the sphere, passes through the sphere at a distance of 12 ft from the center. Find the radius of the circular intersection.

$r = 16$ ft



b. Find the circumference and area of the circular intersection of the plane in Part a.

$C = 2\pi r = 32\pi$ ft, or 100.53 ft
$A = 256\pi$ ft^2, or 804.25 ft^2

Cavalieri's Kitchen is a popular restaurant with fishermen. Every day Greta's staff prepares lunches for fishermen to take on their charter fishing trips down the nearby river. Not only is the food delicious, but the fishermen appreciate that the lunches are packed by Cavalieri's Kitchen in waterproof, spherical-shaped containers. The fishermen tie the containers to their boats and allow them to float along in the water. The food and drinks inside the containers stay cold because the river's primary sources are spring water and melting snow.

The formula for calculating the surface area of a sphere is $A = 4\pi r^2$, where r represents the radius of the sphere.

1. Calculate the surface area for one of the spherical containers if the radius is 8 inches.

$A = 4\pi r^2$
$A = 4\pi(8)^2$
$A = 256\pi$ in^2

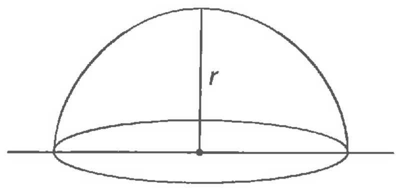
2. **Attend to precision.** Determine the radius of a sphere whose surface area is 1256.64 cm^2. Round your answer to the nearest centimeter.

$A = 4\pi r^2$
$1256.64 = 4\pi r^2$
$r = 10$cm

Lesson 36-1
Surface Area of Spheres

My Notes

Each evening, the fishermen return the empty containers to Cavalieri's Kitchen and typically stay for dinner. The restaurant is an interesting structure, as it is a hemisphere on a wooden platform.



MATH TERMS

When a plane passes through the center of a sphere, the intersection is known as a **great circle**.

3. **Make use of structure.** Describe where a plane would have to intersect a sphere in order to form two hemispheres. Explain your answer.

The plane would have to intersect the sphere in a great circle. That's the only cross section that contains the center of the sphere. Both hemispheres must contain the center.

4. Write a formula for calculating the surface area of the curved surface of the restaurant in terms of its radius, *r*.

$A = 2\pi r^2$ (surface area of the curved part of a hemisphere)

5. Calculate the surface area of the curved surface of the hemisphere with radius 10 feet. Do not include the area of the floor.

$A = 2\pi r^2$

$A = 2\pi(10)^2$

$A = 200\pi^2$ ft, or 628.319 ft^2

CONNECT TO AP

In calculus, you will learn how to derive the formula used to calculate the surface area of a sphere using limits or a definite integral.

6. Use the formula from Item 4 to verify the formula for the surface area of a sphere in terms of its radius, *r*.

$A = 2(2\pi r^2) = 4\pi r^2$

My Notes

7) circle

8) area of each quadrant $= \pi r^2$
since $\frac{4\pi r^2}{4} = \pi r^2$

9) circle

10) $A = 2\pi r^2 + \pi r^2 = 3\pi r^2$

11) $A = 144\pi \text{ cm}^2 \approx 452.39 \text{ cm}^2$

12) $A = 3\pi r^2$
$A = 3\pi(9)^2 = 243\pi \text{ ft}^2$
$\approx 763.41 \text{ ft}^2$

13) $A = 4\pi r^2$
$= 4\pi(7.5)^2$
$= 900\pi \text{ in}^2$

14) $144 = 4\pi r^2$
$r = 6 \text{ ft}$

15a) 30 cm

15b) $225\pi \text{ cm}^2$

16) 14 inches $196 = 4\pi r^2$
$r = 7$

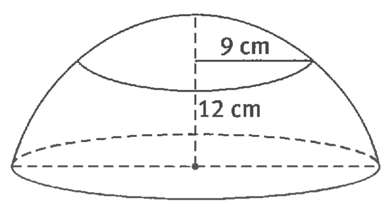
17) $28{,}952.9179 \text{ cm}^2$

Check Your Understanding

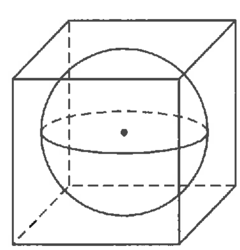
7. Name the shape of the intersection when a plane intersects a sphere in more than one point.
8. A sphere can be divided into four equal quadrants. What is the surface area of each quadrant? Explain how you determined your answer.
9. Name the figure that forms the base of a hemisphere.
10. Write a formula for computing the total surface area of a hemisphere, including the base.
11. Calculate the surface area of a sphere whose radius is 6 cm.
12. A tent is made in the shape of a hemisphere with an 18 ft diameter. How much fabric is used to make the tent, including the floor?

LESSON 36-1 PRACTICE

13. Find the surface area of a sphere, in terms of π, given a radius of 15 in.
14. The surface area of a sphere is $144\pi \text{ ft}^2$. What is the radius?
15. A plane, parallel to the center of the sphere, passes through the sphere at a distance of 12 cm from the center. The radius of the circular intersection is 9 cm.
    a. Find the diameter of the sphere.
    b. What is the area of the great circle?



16. A sphere is inscribed in a cube. The surface area of the sphere is $196\pi \text{ in.}^2$ What is the measure of the edge of the cube?



17. **Make sense of problems.** Cavalieri's Kitchen makes fresh orange juice each morning. The 64 oranges that Greta's staff peels each morning are spherical in shape. The radius of each orange is approximately 6 cm. The orange peels are discarded in a compost pile. How many square centimeters of orange peel are discarded each morning?

My Notes

Volume of a sphere

$V=\frac{4}{3}\pi r^3$

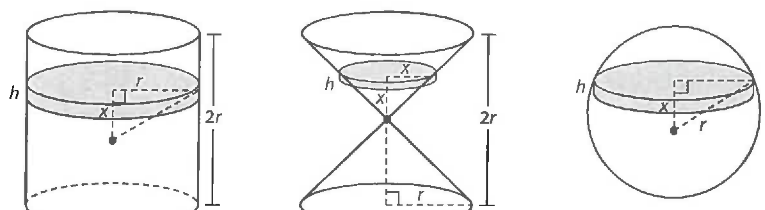
Learning Targets:

- Develop the formula for the volume of a sphere.
- Solve problems by finding the volume of a sphere.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations, Visualization

Greta is promoting a new menu item called the Blueberry Bubble. She tells her morning customers, "The hemisphere of blueberry pancakes that I am creating will absolutely melt in your mouth!" One of the fishermen, a regular morning customer, asks Greta to explain how the pancakes will follow the same mathematical principles as her other famous menu items. She states that, like a cone, a hemisphere is simply a fractional portion of a cylinder.

Greta uses three different stacks of vanilla pancakes to show her customers, placing one blueberry pancake in each stack, as shown below.



1. Describe the shape of each blueberry pancake.

They are cylinders

2. Write the volume formula for the shape identified in Item 1.

$V=\pi r^2 h$

3. How far is each pancake above the center of the corresponding figure?

They are at a distance x above the center of the figure.

4. Write expressions for the cylinder stack.

a. Write an expression for the volume of the single blueberry pancake in the cylinder stack.

$V=\pi r^2 h$

b. Reason abstractly. Write an expression for the *number* of blueberry pancakes that will fill the entire cylinder.

The number that will fill the cylinder is $\frac{2r}{h}$

c. Use parts a and b to write an expression for the total volume of blueberry pancakes that will fill the cylinder stack.

The volume of blueberry pancakes to fill the stack is $\pi r^2 h \cdot \frac{2r}{h}$

My Notes

MATH TIP

Previously, you learned that the volume of a cone is one-third the volume of a cylinder with the same base and height.

5. Notice that the radius of the blueberry pancake in the double cone is x. Write an expression for the volume of the single blueberry pancake in the double cone stack.

The volume is $\pi x^2 h$

6. The volume of the single blueberry pancake in the sphere is equal to the volume of the one in the cylinder minus the volume of the one in the double cone. Write an expression for the volume of the single blueberry pancake in the sphere.

The volume in the sphere is $\pi r^2 h - \pi x^2 h$

7. Construct viable arguments. Show that the relationship you identified in Item 6 holds true for the volumes of the entire stacks of pancakes in each figure as well.

$$V_{sphere} = \pi r^2 h \cdot \frac{2r}{h} - 2\left(\frac{1}{3}\pi r^2 \cdot r\right)$$
$$= 2\pi r^3 - \frac{2}{3}\pi r^3$$
$$= \frac{4}{3}\pi r^3$$

8. Write the formula for the volume of a sphere.

$V = \frac{4}{3}\pi r^3$

9. What is the formula for the volume of a hemisphere?

Volume of a hemisphere $= \frac{2}{3}\pi r^3$

10. The Blueberry Bubble will have a base pancake with a radius of 3 inches, and each of the six pancakes in the hemisphere will have a thickness of 0.5 inch.

a. Draw a diagram for the Blueberry Bubble.

b. What is the volume of the Blueberry Bubble?

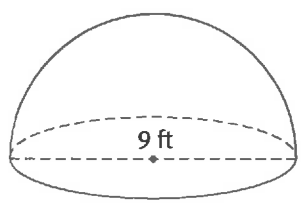
18π in^3

Check Your Understanding

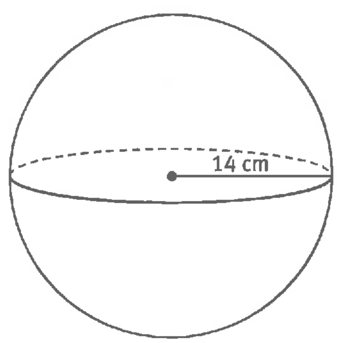
11. Suppose a sphere is divided into four equal quadrants. Write a formula to find the volume for $\frac{1}{4}$ of a sphere.
12. In terms of units of measure, explain why the surface area formula for a sphere has a squared radius and the volume formula for a sphere has a cubed radius.
13. The diameter of a sphere is 8 in. What is the volume of the sphere?
14. A sphere has a surface area of approximately 314 in.^{2} What is the volume of the sphere to the nearest cubic inch?

LESSON 36-2 PRACTICE

15. Find the volume of the figure shown, to the nearest hundredth.



16. Determine the volume of the sphere shown below, to the nearest tenth.



17. **Construct viable arguments.** Four golf balls, each with radius 1 inch, fit snugly into a 2 in. × 2 in. × 8 in. box. Is the total volume left over inside the box greater than or less than the volume of a fifth golf ball? Justify your answer.
18. A spherical scoop of ice cream has been placed on top of a cone, as shown. If the ice cream melts completely, will it overflow the cone? Justify your answer.

The ice cream will overflow the cone.

Volume of ice cream $= 36\pi$ in^3
Volume of cone $= 5\frac{1}{3}\pi$ in^3

My Notes

11) $V = \frac{1}{3}\pi r^3$

12) The surface area of a sphere describes the number of square units necessary to cover the entire surface of the sphere. The volume of the sphere describes the number of cubic units that can fit inside the sphere.

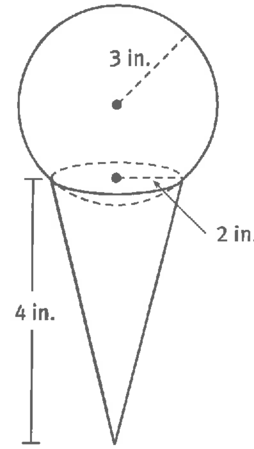
13) $\frac{256}{3}\pi$ in^3 ≈ 268.08 in^3

14) $314 = 4\pi r^2$
$V = 523$ in^3

15) $V = \frac{4}{3}\pi(4.5)^3$
$= 190.85$ ft^3

16) $V = \frac{4}{3}\pi(14)^3 = 11494$ cm^3

17) $V = 32$ in^3; The volume of the four golf balls is slightly less than 17 cubic inches. There is room for more golf balls since each is about 4.2 cubic inches.



My Notes

Learning Targets:

- Compare parallelism in Euclidean and spherical geometries.
- Compare triangles in Euclidean and spherical geometries.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Visualization, Create Representations, Look for a Pattern, Use Manipulatives

Greta has used various relationships among geometric figures such as lines, angles, polygons, circles, and three-dimensional objects to create interest in mathematics at her restaurant. She, like many others, has relied on an axiomatic system as the basis of many geometric theorems; Euclidean properties are sufficient for most real-world geometric applications. Greta does realize, however, that the study of mathematics goes well beyond Euclidean geometry—as the world, unlike Greta's pancakes, is not flat.

An understanding of spherical geometry is what allowed NASA engineers to perform the necessary calculations to get the rover to land on Mars in May 2008. Mathematicians, and curious folks like Greta, have pondered the existence of non-Euclidean geometries since the early 1800s. Research in these areas continues to be a considerable focus of study today.

Spherical geometry is one branch in the non-Euclidean field. It explores the geometric characteristics of figures on the surface of a sphere. There are significant differences in properties with this change in perspective. To understand the concepts related to spherical geometry, it is helpful to have a globe or a ball to contemplate these ideas.

1. **Use appropriate tools strategically.** Choose a point on the sphere provided by your teacher. Imagine that an ant walks on the sphere in a straight line so that it does not fall off.
   a. Trace the path that the ant would follow.
   b. Describe the path walked by the ant.
   c. Use a rubber band or masking tape to mark the path of the ant.

   The path traces a great circle of the sphere.

2. Choose a different point on the sphere. Imagine that a second ant walks in a straight line, trying to avoid the path of the first ant. Mark the path of the second ant.

   The second ant creates a second great circle intersecting the first.

3. How can the second ant choose a path that will not cross the path of the first ant?

   It is not possible for the second ant to choose a path that doesn't cross the first.

MATH TIP

Great circles are the largest possible circles drawn on a sphere.

My Notes

4. In Euclidean geometry, given a line and a point not on the line, there is exactly one line through the point parallel to the given line. Explain why no lines are parallel in spherical geometry.

In spherical geometry, no such parallel line exists.

5. In Euclidean geometry, if two lines intersect, they intersect in exactly one point. In spherical geometry, what is true about any two lines?

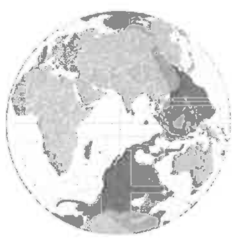
In spherical geometry, two distinct (not concurrent) lines intersect in exactly two points.

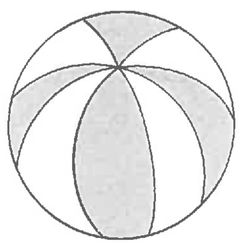
6. In spherical geometry, a **line** is defined to be a **great circle** of the sphere. Lines of longitude on a globe are examples of these. Why do you suppose that airlines design flight paths along spherical lines?

Spherical lines define the shortest path between two points on a sphere.

Antipodal points (also called antipodes) are two points that are diametrically opposite. A straight line through the center of the Earth connects them.

7. Locate the antipode for your city. State its location.





8. **Construct viable arguments.** Explain why most points on Earth have oceanic antipodes.

Approximately 71% of the earth's surface is covered with water, most of which is ocean.

9. Choose a point on your sphere.

a. Draw two great circles through the point.

b. Use a protractor to measure the angles formed at the given point on the sphere by the great circles.

The angles add up to 360°; the opposite (vertical) angles have equal measures.

c. What do you notice about the measures of the angles formed by the great circles?

Each pair of angles formed at opposite ends of a lune is congruent. Opposite angles at the point where the great circles intersect are congruent.

My Notes

MATH TERMS

A **lune** is a region bounded by two great circles. The word is derived from the Latin word for *moon*.

10. Model with mathematics. Take a sphere. Choose a point N on your sphere.

a. Draw a great circle through point N.

b. Draw a second great circle through point N to create a ***lune*** with a 90° angle.

c. Draw a great circle that does not contain point N to divide the lune into two congruent regions.

d. How many nonoverlapping sections are created by the intersections of the three great circles?

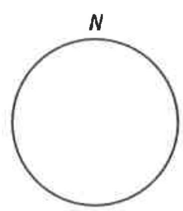
Eight nonoverlapping regions are formed.

e. Each section is a spherical triangle. Use a protractor to measure the three angles of a triangle.

All three angles are 90°

f. What is the sum of the measures of those three angles?

The sum of the measures are 270°



11. Choose any three random noncollinear points on the sphere. Draw the triangle through the points. Remember to draw the segments along a great circle.

A spherical triangle should be shown.

12. Measure the three angles of the triangle and determine the sum of their measures.

The sum will be greater than 180°.

My Notes

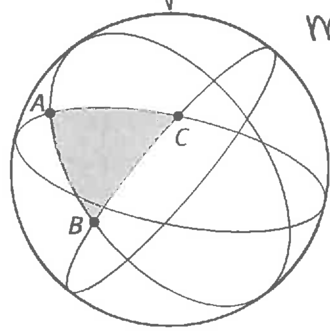
13. Use the given diagram with triangle ABC.

a. What happens to the sum of the measures of the angles in the triangle as points B and C come closer and closer together?

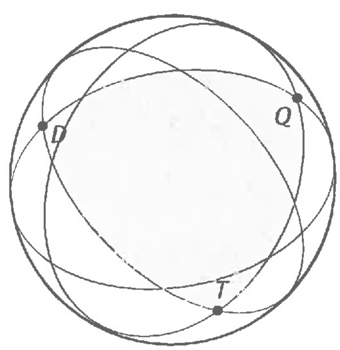
As points B and C come closer and closer together, $m\angle A$ approaches $0°$, and $m\angle B$ and $m\angle C$ approach $90°$.

b. Based on your answer in Part a, what is the least possible sum for the three angles of a spherical triangle?

The least possible sum for the three angles of a spherical triangle is a measure slightly greater than 180°.



14. Use the given diagram with triangle DQT.



a. What happens to the measure of each angle in the triangle as the points move outward and approach the same great circle?

As the points move outward and approach the same great circle, the measures of each of the triangles approach 180°.

b. Based on your answer in part a, what is the greatest possible sum for the three angles of a spherical triangle?

The greatest possible sum for the three angles of a spherical triangle is a measure slightly less than 540°.

15. Write the possible measurements for the sum of the measures of the angles of a triangle as a compound inequality.

$180° < x < 540°$

My Notes

16) The lines of the Earth's latitude are not considered lines in spherical geometry since they are not great circles.

17) Example: from Wellington, New Zealand to just north of Madrid, Spain.

18) 2 points of intersection
4 angles

19) Helps to develop an axiomatic system and develops logical reasoning based on accepted premises. The area, volume and measurement formulas are useful

20) In Euclidean geometry, the sum of the measures of a triangle is 180°. In spherical geometry, it varies between 180° and 540°. In spherical geometry, two lines cannot be parallel. In spherical geometry, two lines always intersect at two points.

Check Your Understanding

16. Are the lines of the Earth's latitude considered lines in spherical geometry? Explain.

17. Locate two land-based antipodes on Earth.

18. Refer to Item 9. When two great circles intersect, how many points of intersection result? How many angles are formed at each point of intersection along with four lunes?

LESSON 36-3 PRACTICE

19. Why do you suppose Euclidean geometry is still considered to be an essential component of the high school curriculum?

20. **Make use of structure.** Compare and contrast Euclidean geometry and spherical geometry.